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Optimal Control Synthesis of Epidemic Model

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Abstract— **The spread of a virus or the outbreak of an epidemic are natural examples of stochastic processes. Understanding the epidemic dynamics, and finding out efficient techniques to control it, is a challenging issue. This paper investigates the optimal use of intervention strategies to mitigate the spread of infectious diseases. Classical mathematical descriptions of such phenomenon include various branching processes such as the SIR (Susceptible-Infected-Recovered). One reason for mathematical modelling is to analyze and predict the extent of emerging diseases and develop proposed control measures. The quadratic regulator format was used to formulate the optimal control problem, and two different optimal control techniques were investigated: Single Network Adaptive Critic (SNAC), which is a direct application of reinforcement learning theory to the optimality necessary conditions, and Approximate Sequence Riccati Equation (ASRE), which is a global optimal feedback control technique for general nonlinear systems with nonquadratic performance criteria. According to the results obtained during simulations, we claim that the proposed model and control strategy can be considered a good candidate to study viral spreading in the world. Also, the result shows that the Single Network Adaptive Critic is more accurate than the Approximate Sequence Riccati Equation.**

Index Terms: **ASRE, Epidemic, SNAC, SIR, Optimal control.**

I. INTRODUCTION

Through the last century, there have been significant decreases in the morbidity and mortality of many infectious diseases due to the introduction of medicines and vaccines, as well as better living conditions, including access to healthcare and surveillance systems. However, both in developing and developed nations, infectious diseases continue to be major sources of human suffering and death. A grasp of the epidemiological features of a disease is essential for the effective implementation of infectious disease control or prevention methods [1]. The process of constructing models requires an awareness of several facets of infectious diseases, such as the clinical and biological understanding of the infection agent. Additionally, research on the effects of earlier outbreaks of infectious illnesses on our health system is necessary for a deeper comprehension of their behavior [2][3].

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Infectious diseases are responsible for significant health and economic problem in society. Mathematical modelsuseassumptionsandstatisticalinferencesindetermini ngparameters for the spread of diseases. Mathematical models in recent years have been used to guide policy makers responding to the emergency of the diseases including Measles [4][5][6], H1N1 influenza [7][8], Hepatitis C Virus (HCV) [9][10], Whooping cough [11], HIV [12][13], Ebola [14][15][16], Coronavirus [17][18] and many others. The behavior of the infectious disease is investigated and controlled using an optimal control strategy. The aim is to optimize the overall process, minimizing the infected people, and maximizing the recovery process.

II. MATHEMATICAL MODEL OF SIR

In this paper, we studied the stochastic SIR epidemic model on complex networks. The stochastic model studied captured the randomness in disease transmission observed in a real-life epidemic which serves as a model to influence the outcome of an emerging epidemic. The SIR model, created by Kermack and McKendrick, is a straightforward representation of an infectious disease epidemic in a sizable population. The letters S, I, and R represent the numbers of the three sorts of people we believe make up the population (which is why this is called an SIR model). These are all time-dependent functions that vary in accordance with a set of differential equations. The SIR model is shown in Figure. 1.

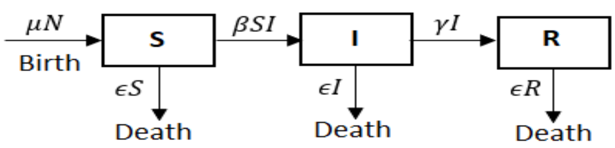

Figure. 1 . SIR model without control

In the epidemiological SIR model here, the following assumptions are made [19]:

- The way a person can leave the susceptible group (S), is to become infected, and the way a person can leave the infected group (I), is to recover from the disease.
- Any recovered person in (R) has permanent immunity.
- The size of the population is (N), where is a variable number and large, where N(t) = S(t) + I(t) + R(t).

- The spreading rate (β) knows as the transmission rate. and the recovery rate (γ), is the same for all individuals and is supposed positive.

The model is a demography system i.e. birth rate, and death rate (taken into consideration).

The equations of the SIR model are [19][20]:

$$\frac{dS}{dt} = \mu N - \frac{\beta SI}{N} - \epsilon S \tag{1}$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - \epsilon I \tag{2}$$

$$\frac{dR}{dt} = \gamma I - \epsilon R \tag{3}$$

The size of population (N) is a variable and large, Where:

$$N(t) = S(t) + I(t) + R(t) \tag{4}$$

μN: with the positive sign is the fraction of total population, in other words is the birth rate.

ϵS, ϵI, and ϵR: with a negative sign, is the fraction of susceptible, infected, and recovery people, in other word is the death rate of mortality of individuals.

βSI: represents the number of newly infected individuals per unit time corresponds to homogeneous mixing of the infected and susceptible classes.

γI: represents the portion of the infected individuals that recovering.

The spreading rate $\beta > 0$ is ratio,

$$\beta = \tau c \tag{5}$$

Where τ is the transmissibility (probability of infection given to contact between a susceptible and infected individual), c is the average rate of contact between susceptible and infected, but in, general the spreading rate is hard to predict.

The recovery rate is $\gamma > 0$ can be measured in a laboratory, for, example if people, on average, stay sick for two days the recovery rate is γ = 1/2, so recovery rate, in general is [19],

$$\gamma = \frac{1}{D} \tag{6}$$

Where D is duration of disease for those recovered.

III. OPTIMAL CONTROL THEORY

A contemporary development of the calculus of variations, the theory of optimal control has found several uses in a variety of scientific disciplines, particularly in epidemiology with regard to the detection and treatment of disease [20].

A. Quadratic Regulator Problem

The Quadratic Regulator Problem (QRP) is the most famous type of the optimal control problem, where the aim is to minimize the total energy expenditure $\boldsymbol{J}$ of both the system states and control actions, which are weighted using the weighting matrices of $\boldsymbol{Q(t)}$ and $\boldsymbol{R(t)}$ respectively. In addition to minimizing the energy expenditure, the system is also required to satisfy the imposed constrains on the system, which come in two types, the first type is the equality constrains which are represented by the dynamic state constrains of the system, while the second type is the inequality constrains which often represent the physical limits on both the system states $\boldsymbol{x(t)}$ and control actions$\boldsymbol{u(t)}$, in mathematical form the QRP can be formulated as:

$$\underset{\boldsymbol{u}(t)}{minimize} \; J(\boldsymbol{u}) = \frac{1}{2}\cdot \int_0^{\infty} [\boldsymbol{x}(t)^{\,T} \cdot \boldsymbol{Q}(t)\cdot \boldsymbol{x}(t) + \boldsymbol{u}(t)^T \cdot \boldsymbol{R}(t)\cdot \boldsymbol{u}(t)]\cdot dt \tag{7}$$

$$\left.\begin{aligned} subject\ to: \dot{\boldsymbol{x}}(t) = \boldsymbol{a}\big(\boldsymbol{x}(k), \boldsymbol{u}(k)\big), \boldsymbol{x}(0) = \boldsymbol{x_0} \\ \boldsymbol{x}_l \le \boldsymbol{x}(t) \le \boldsymbol{x}_u \\ \boldsymbol{u}_l \le \boldsymbol{u}(t) \le \boldsymbol{u}_u \end{aligned}\right\} \forall\, t$$

The state inequality constrains are often handled separately using either the valentine transformation or the penalty function technique , on the other hand the control inequality constrains are handled through pontryagin's minimum principle (PMP) .

The necessary conditions for optimality for the QRP are derived from the Hamiltonian function which follows from the theory of Lagrange multipliers $\boldsymbol{\lambda(t)}$, the Hamiltonian function is formulated as:

$$H\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)\big) = \frac{1}{2}\cdot \boldsymbol{x}(t)^T \cdot \boldsymbol{Q}(t)\cdot \boldsymbol{x}(t) + \frac{1}{2}\cdot \boldsymbol{u}(t)^T \cdot \boldsymbol{R}(t)\cdot \boldsymbol{u}(t) + \boldsymbol{\lambda}(t)^T \cdot \boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{8}$$

The necessary conditions follow as (corollary):

State Equations:

$$\dot{\boldsymbol{x}}(t) = \frac{\partial H(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{\lambda}(t))}{\partial \boldsymbol{\lambda}(t)} = \boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{9}$$

Costate Equations:

$$\begin{aligned} \dot{\boldsymbol{\lambda}}(t) &= -\frac{\partial H\big(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{\lambda}(t)\big)}{\partial \boldsymbol{x}(t)} \\ &= -\boldsymbol{Q}(k)\cdot \boldsymbol{x}(k) - \left[\frac{\partial \boldsymbol{a}\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big)}{\partial \boldsymbol{x}(t)}\right]^T \cdot \boldsymbol{\lambda}(t) \end{aligned} \tag{10}$$

Optimal Control Equations:

$$\frac{\partial H\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)\big)}{\partial \boldsymbol{u}(t)} = \boldsymbol{0} \rightarrow$$

$$\boldsymbol{u}(t) = -\boldsymbol{R^{-1}}\,(t)\cdot \left[\frac{\partial \boldsymbol{a}(\boldsymbol{x}(t),\boldsymbol{u}(t))}{\partial \boldsymbol{u}(t)}\right]^T \cdot \boldsymbol{\lambda}(t) \tag{11}$$

Boundary conditions:

$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\boldsymbol{\lambda}(\infty) = \boldsymbol{\lambda}_f$$

To handle the control inequality constrains, the PMP is applied as follows:

$$\boldsymbol{H(x(t), u^*(t), \lambda(t), t) \le H(x(t), u(t), \lambda(t), t)}$$

If the QRP was applied in a discrete closed loop fashion, the PMP inequality relation simply becomes a saturation limiter as follows:

$$\boldsymbol{u}(k) = \begin{cases} \boldsymbol{u}_l & if\ \overline{\boldsymbol{u}}(k) < \boldsymbol{u}_l \\ \overline{\boldsymbol{u}}(k) & otherwise \\ \boldsymbol{u}_u & if\ \overline{\boldsymbol{u}}(k) > \boldsymbol{u}_u \end{cases}$$

Where $\overline{\boldsymbol{u}}(\boldsymbol{k})$: is the optimal control at the discrete time step $\boldsymbol{k}$, before applying the PMP.

The weighing matrices $\boldsymbol{Q(t)}$ and $\boldsymbol{R(t)}$ are rarely chosen to be an independent function of time, however they often chosen to be function of the system states $\boldsymbol{x(t)}$ making the optimal control problem more stable and robust to system changes i.e.,

$$\boldsymbol{Q}(t) \rightarrow \boldsymbol{Q}\big(\boldsymbol{x}(t)\big), \boldsymbol{R}(t) \rightarrow \boldsymbol{R}\big(\boldsymbol{x}(t)\big).$$

Although the necessary conditions for optimality of the optimal control equation reduced the task of the optimal control problem to just solving the state and costate equations while satisfying the optimal control equations, however due to the inherent nonlinearity in these equations and the split in boundary conditions (which is often referred to as the curse of complexity), these equations are often solved using very complex solvers.

In this paper two different closed loop techniques are used to solve this problem namely: The Approximate Sequence Ricatti Equation (ASRE), and The Single Network Adaptive Critic (SNAC) [21][22][23].

B. Approximate Sequence Riccati Equation (ASRE)

This technique starts by formulating the system in the following state dependent nonaffine continuous differential form:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{x}(\boldsymbol{t})) \cdot \boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})) \cdot \boldsymbol{u}(\boldsymbol{t}) \quad (13)$$

The latter form uses the State dependent coefficient (SDC) parameterization, which is the process of transforming a nonlinear system into linear like structure, it is termed nonaffine since the $\boldsymbol{B}$ matrix is a function of the control signals $\boldsymbol{u}(\boldsymbol{t})$, unlike other techniques (such as State Dependent Riccati Equation (SDRE)) which requires that the $\boldsymbol{B}$ matrix to be affine i.e., $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}(\boldsymbol{k}))$.

The technique then defines a sequence of riccati equations which are arbitrary close to the true system, where each equation (except the first one) is a direct application of the extended linearization theory on the linear quadratic regulator (LQR) of time variant systems, these equations are then solved sequentially, where the solution of one riccati equation is the starting solution of the next riccati equation.

The ASRE methodology is formulated as follows:

The system dynamic (state) equations at the $\boldsymbol{0^{th}}$ iteration takes the following form:

$$\dot{\boldsymbol{x}}^{[0]} = \boldsymbol{A}(\boldsymbol{x}_0) \cdot \boldsymbol{x}^{[0]} + \boldsymbol{B}(\boldsymbol{x}_0, \boldsymbol{0}) \cdot \boldsymbol{u}^{[0]} \quad (14)$$

And for the i^{th} iteration, where $i > 0$, takes the following form:

$$\dot{\boldsymbol{x}}^{[i]} = \boldsymbol{A}(\boldsymbol{x}^{[i-1]}) \cdot \boldsymbol{x}^{[i]} + \boldsymbol{B}(\boldsymbol{x}^{[i-1]}, \boldsymbol{u}^{[i-1]}) \cdot \boldsymbol{u}^{[i]} \quad (15)$$

And the initial conditions for all the iterations are:

$$\boldsymbol{x}^{[i]}(t_0) = \boldsymbol{x}_0$$

The riccati equation at the $\boldsymbol{i^{th}}$ iteration takes the following form (note that the time dependency was drop for convince):

$$\boldsymbol{P}^{[i]} \cdot \boldsymbol{A}\left(\boldsymbol{x}^{[i-1]}\right) + \boldsymbol{A}^T\left(\boldsymbol{x}^{[i-1]}\right) \cdot \boldsymbol{P}^{[i]} - \boldsymbol{P}^{[i]} \cdot \boldsymbol{B}\left(\boldsymbol{x}^{[i-1]}, \boldsymbol{u}^{[i-1]}\right) \cdot \boldsymbol{R}^{-1}\left(\boldsymbol{x}^{[i-1]}, \boldsymbol{u}^{[i-1]}\right) \cdot \boldsymbol{B}^T\left(\boldsymbol{x}^{[i-1]}, \boldsymbol{u}^{[i-1]}\right)\boldsymbol{P}^{[i]} + \boldsymbol{Q}(\boldsymbol{x}^{[i-1]}) = 0 \quad (16)$$

The $\boldsymbol{P}$ matrix (which describes the relation between the costate and state variables i.e., $\boldsymbol{\lambda}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) \cdot \boldsymbol{x}$) is the only unknown in this equation, and it is used to calculate the control signal at the $\boldsymbol{i^{th}}$ iteration as follows:

$$\boldsymbol{u}^{[i]} = -\boldsymbol{R}^{-1}(\boldsymbol{x}^{[i-1]}) \cdot \boldsymbol{B}^T(\boldsymbol{x}^{[i-1]}, \boldsymbol{u}^{[i-1]}) \cdot \boldsymbol{P}^{[i]} \cdot \boldsymbol{x}^{[i]}$$

This iteration process is repeated until the following convergence criterion is satisfied:

$$\lim_{i\to\infty}\left\|\boldsymbol{x}^{[i]} - \boldsymbol{x}^{[i-1]}\right\| < \varepsilon \quad (17)$$

Where: ε is some specified tolerance [24][25].

C. Single Network Adaptive Critic (SNAC)

Dynamic programming provides the most intuitive and comprehensive solution to general optimal control problems in a state feedback form, however dynamic programming relies fundamentally on solving the Hamilton Bellman equation, unfortunately the computational demands of this equation grow exponentially with the complexity of the problem.

Fortunately, Werbos in 1992 get around this problem using the concept of Approximate dynamic programming (ADP). One famous way to implement the ADP is through the Action Critic network (AC), this network compromises of two networks namely Action, and Critic. The Action network learns to find the relation between the states $\boldsymbol{x}$ of the system and the control action $\boldsymbol{u}$ at the same discrete time step $\boldsymbol{k}$, while the Critic network learns to find the relation between the system states $\boldsymbol{x}$ and the costates $\boldsymbol{\lambda}$ at the same discrete time step $\boldsymbol{k}$, this learning process is achieved using the Dual Heuristic Programming (DHP), where the necessary conditions for optimality are used heuristically to train these networks. A special type of the AC network arises when the system model is affine in the control signals, this type is called Single Network Adaptive Critic (SNAC). Since the control signals $\boldsymbol{u}$ are affine, then only the states and costate variables are needed in the optimal control equation and hence the action network is no longer required. The SNAC technique trains the critic network to achieve the relation between the states $\boldsymbol{x}$ at time step $\boldsymbol{k}$ and the costates $\boldsymbol{\lambda}$ at the next time step $\boldsymbol{k+1}$, the training procedure of the SNAC technique consists of two parts:

Pretraining: before straining with the actual procedure of the SNAC technique, the critic network is pretrained with a stabilizing control rule first, which may be hard to find, however the authors in found that the LQR technique applied on linear model of the system does the job (especially when the desired equilibrium point is stable).

The training procedure of the SNAC technique is as follows:

1. Generate P random states vectors $\boldsymbol{x}_{k,1}, \boldsymbol{x}_{k,2}, \dots, \boldsymbol{x}_{k,P}$, in their admissible region.
2. Feed these states into the critic network, to get the estimated next costate vectors $\hat{\boldsymbol{\lambda}}_{k+1,1}, \hat{\boldsymbol{\lambda}}_{k+1,2}, \dots, \hat{\boldsymbol{\lambda}}_{k+1,P}$.
3. The resultant next costate vectors along with the state vectors are fed to the optimal control equation to get the following control vectors $\boldsymbol{u}_{k,1}, \boldsymbol{u}_{k,2}, \dots, \boldsymbol{u}_{k,P}$.
4. The resultant control vectors along with the current states are fed to the system model to obtain the next states $\boldsymbol{x}_{k+1,1}, \boldsymbol{x}_{k+1,2}, \dots, \boldsymbol{x}_{k+1,P}$
5. The resultant next states vectors are fed to the critic network to get the estimated next-next costate vectors of $\hat{\boldsymbol{\lambda}}_{k+2,1}, \hat{\boldsymbol{\lambda}}_{k+2,2}, \dots, \hat{\boldsymbol{\lambda}}_{k+2,P}$. (Note that the critic network gives the costate vector which is one step ahead in time with respect to the input state vector).

6. Feeding the resultant next-next costate vectors along with the next state vectors (resultant from the system model) to the costate equation heuristically results in a better next costate vectors $\boldsymbol{\lambda}_{k+1,1}, \boldsymbol{\lambda}_{k+1,2}, \dots, \boldsymbol{\lambda}_{k+1,P}$ than the ones obtained previously, hence they are used to train the critic network [19][26][27].
7. The SNAC technique is also applied iteratively until the mean squared error $e_{c,i} = \|\boldsymbol{\lambda}_{k+1,i} - \hat{\boldsymbol{\lambda}}_{1+1,i}\|$ (for $i = 1 \dots P$) is less than a specified limit.

IV. OPTIMAL CONTROL PROBLEM FORMULATION

The first step in formulating the optimal control problem is to put the model in the state dependent form of:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \cdot \boldsymbol{u}(t) \quad (18)$$

In order to do this, the equilibrium points of the system must be first be calculated, and then the desired one must be shifted to the origin [19][28][29].

A change of variables is first applied to the system model of with the following substitutions:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ I \\ R \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

The resultant model is:

$$\dot{x_1}(t) = \mu \cdot N - \frac{\beta \cdot x_1 \cdot x_2}{N} - \varepsilon \cdot x_1 - u_2 \cdot x_1 \quad (19)$$

$$\dot{x_2}(t) = \frac{\beta \cdot x_1 \cdot x_2}{N} - \gamma \cdot x_2 - u_1 \cdot x_2 \quad (20)$$

$$\dot{x_3}(t) = \gamma \cdot x_2 - \varepsilon \cdot x_3 + u_2 \cdot x_1 + u_1 \cdot x_2 \quad (21)$$

This system has the following two equilibrium points:

- Dead: where all the population is susceptible to the disease:

$$\begin{bmatrix} S_f \\ I_f \\ R_f \end{bmatrix} = \begin{bmatrix} N \cdot \frac{\mu}{\varepsilon} \\ 0 \\ 0 \end{bmatrix} \quad (22)$$

- Live: where most of final population are recovered from the disease:

$$\begin{bmatrix} x_{1_f} \\ x_{2_f} \\ x_{3_f} \end{bmatrix} = \begin{bmatrix} N \cdot \frac{\varepsilon+\gamma}{\beta} \\ \frac{N \cdot (\beta \cdot \mu - \varepsilon^2 - \gamma \cdot \varepsilon)}{\beta \cdot (\varepsilon+\gamma)} \\ \frac{\gamma \cdot N \cdot (\beta \cdot \mu - \varepsilon^2 - \gamma \cdot \varepsilon)}{\varepsilon \cdot \beta \cdot (\varepsilon+\gamma)} \end{bmatrix} \quad (23)$$

The model is now shifted, such that the Live equilibrium point is now at the origin as follows:

$$\dot{x_1}(t) = -\frac{1}{N \cdot (\varepsilon + \gamma)} \cdot [(\beta \cdot N \cdot \mu + \beta \cdot \gamma \cdot x_2(t) + \beta \cdot \varepsilon \cdot x_2(t)) \cdot x_1(t) + N(\varepsilon + \gamma)^2 \cdot x_2(t)] - u_2(t) \cdot (x_1(t) + x_{1_f})$$

$$\dot{x_2}(t) = \frac{1}{N \cdot (\varepsilon + \gamma)} \cdot [(\beta \cdot N \cdot \mu - N \cdot \varepsilon^2 - N \cdot \varepsilon \cdot \gamma) \cdot x_1(t) + (\beta \cdot \varepsilon + \beta \cdot \gamma) \cdot x_2(t) \cdot x_1(t)] - u_1(t) \cdot (x_2(t) + x_{2_f})$$

$$\dot{x_3}(t) = \gamma \cdot x_2(t) - \varepsilon \cdot x_3(t) + u_2(t) \cdot \left(x_1(t) + x_{1_f}\right) + u_1(t) \cdot \left(x_2(t) + x_{2_f}\right)$$

Since the model is inherently affine in the control signals u, it can be formulated in the following simpler state depend form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{x}(t)) \cdot \boldsymbol{u}(t) \quad (24)$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \\ -b_{21} & -b_{12} \end{bmatrix}$$

$$a_{11} = -\frac{(\beta \cdot N \cdot \mu + \beta \cdot \gamma \cdot x_2(t) + \beta \cdot \varepsilon \cdot x_2(t))}{N \cdot (\varepsilon + \gamma)}$$

$$a_{12} = -N \cdot (\varepsilon + \gamma)$$

$$a_{21} = \frac{(\beta \cdot N \cdot \mu - N \cdot \varepsilon^2 - N \cdot \varepsilon \cdot \gamma)}{N \cdot (\varepsilon + \gamma)}$$

$$a_{22} = \frac{\beta \cdot (\varepsilon + \gamma) \cdot x_1(t)}{N \cdot (\varepsilon + \gamma)}$$

$$a_{32} = \gamma$$

$$a_{33} = -\varepsilon$$

$$b_{12} = -\left(x_1(t) + x_{1_f}\right)$$

$$b_{21} = -\left(x_2(t) + x_{2_f}\right)$$

Two different optimization criteria were applied, in the first criterion the $\boldsymbol{Q}, \boldsymbol{R}$ were chosen to be constant diagonal matrices as:

$$\boldsymbol{Q} = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$$

The costate model and the optimal control equation of the current system can be formulated as:

The costate model [19][30][31][32]:

$$\dot{\boldsymbol{\lambda}}(t) = -\boldsymbol{Q} \cdot \boldsymbol{x}(t) - \left[\frac{\partial \boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t))}{\partial \boldsymbol{x}(t)}\right]^T \cdot \boldsymbol{\lambda}(t) \quad (25)$$

$$\dot{\lambda_1} = -q_{11} \cdot x_1 + \left(\frac{(\beta \cdot N \cdot \mu + \beta \cdot x_2 \cdot \gamma + \beta \cdot x_2 \cdot \varepsilon)}{N \cdot (\varepsilon + \gamma)} + u_2\right) \cdot \lambda_1 - \left(\frac{(\beta \cdot N \cdot \mu - N \cdot \varepsilon^2 - N \cdot \varepsilon \cdot \gamma + \beta \cdot (\varepsilon + \gamma) \cdot x_2)}{N \cdot (\varepsilon + \gamma)}\right) \cdot \lambda_2 - u_2 \cdot \lambda_3$$

$$\dot{\lambda_2} = -q_{22} \cdot x_2 + \left(\frac{(\beta \cdot \gamma \cdot x_1 + \beta \cdot \varepsilon \cdot x_1 + N \cdot (\varepsilon + \gamma)^2)}{N \cdot (\varepsilon + \gamma)} + u_2\right) \cdot \lambda_1 - \left(\frac{\beta \cdot x_1}{N} - u_2\right) \cdot \lambda_2 - (\gamma + u_1) \cdot \lambda_3$$

$$\dot{\lambda_3} = -q_{33} \cdot x_3 + \varepsilon \cdot \lambda_3$$

The Optimal Control Equation:

$$\boldsymbol{u}(t) = -\boldsymbol{R}(t)^{-1} \cdot \left[\frac{\partial \boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t))}{\partial \boldsymbol{u}(t)}\right]^T \cdot \boldsymbol{\lambda}(t) \quad (26)$$

$$u_1(t) = -\frac{1}{r_{11}} \cdot \left(x_2(t) + x_{2_f}\right) \cdot (\lambda_3(t) - \lambda_2(t))$$

$$u_2(t) = -\frac{1}{r_{22}} \cdot \left(x_1(t) + x_{1_f}\right) \cdot (\lambda_3(t) - \lambda_1(t))$$

V. SIMULATION RESULTS

To justify the impact of optimal control, we have used different control algorithms to solve the optimality system numerically. The simulation which we carried out by using the parametric values given in Table 1. The exact numerical values of all the parameters of the OCP are summarized in Table II. A comparative study between the system with controls and without control has been presented to realize the positive impact of vaccination and treatment in controlling the infectious diseases.

Table 1. Covid-19 Disease data [19]

β	0.47/ (Days. People)
γ	0.343 /Days
μ	0.0007
ϵ	0.0007
T	40 Days
S_0	80% People
I_0	20% People
R_0	0% People

Table 2. Exact values of the OCP parameters

Parameter	**Description**	**Value**
N	Number of sample points	800
h	Sampling period	0.05 [day]
$\boldsymbol{x}_0$	Initial states	$[0.8,0.2,0]^T$
$\boldsymbol{R}$	Control weighing matrix	$diag([1,1])$
$\boldsymbol{Q}$	State weighting matrix	$diag([1, 10, 1])$
$\boldsymbol{u}_u$	Upper limit on control signal	[1,1]
$\boldsymbol{u}_l$	Lower limit on control signal	[0,0]

Figure. 2 show the time series of the susceptible (S), infected (I) and recovered (R) individuals both with and without control. Figure. 3 represent the optimization criterion of applying the ASRE and SNAC techniques. Figure . 4 display the optimal control signals of applying the ASRE and SNAC techniques along with the no control solution. From the simulation result, we see those optimal controls due to vaccination and treatment are very effective for reducing the number of susceptible and infected individuals and so enhancing the number of recovered individuals significantly.

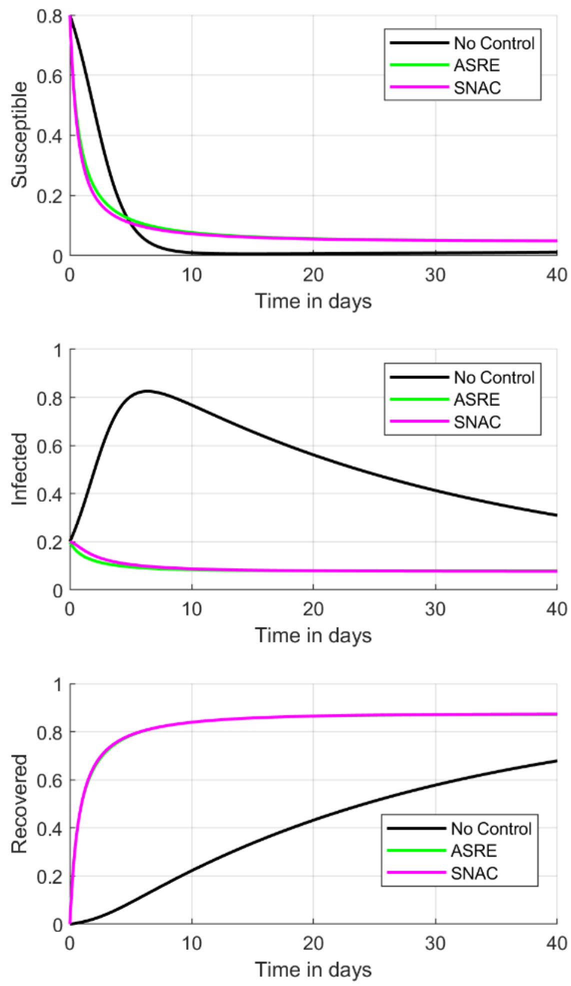

Figure 2. The optimal states of applying the ASRE and SNAC techniques along with the no control solution

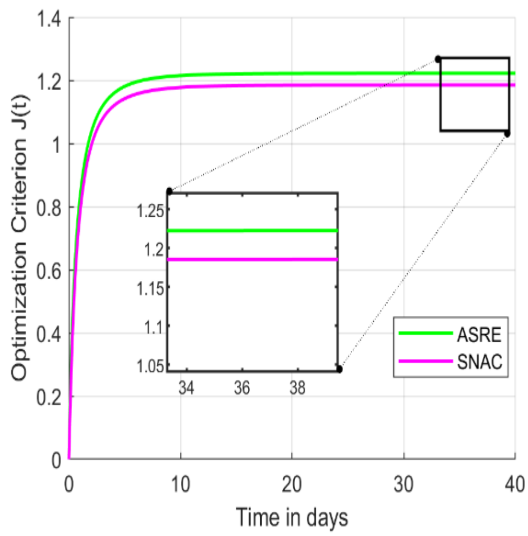

Figure 3. The optimization criterion of applying the ASRE and SNAC techniques

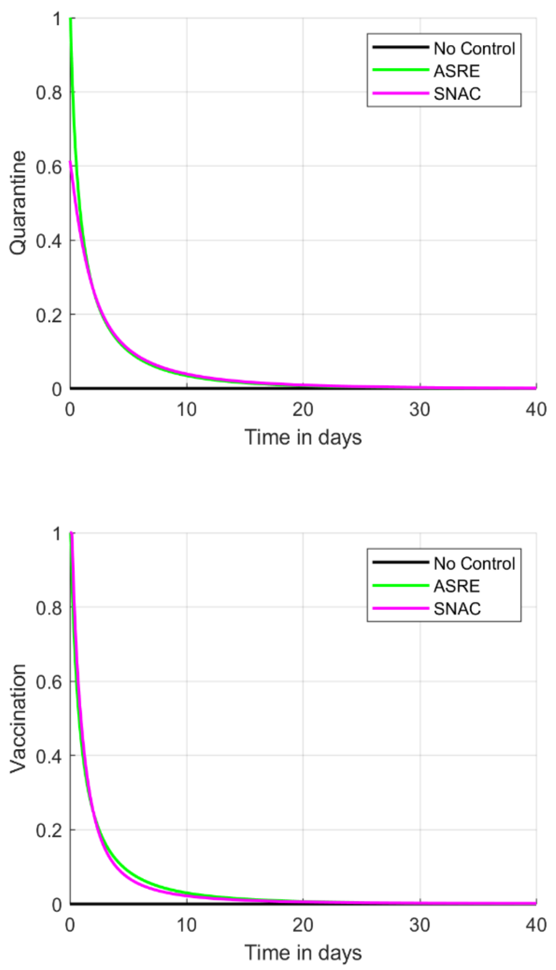

Figure 4. The optimal control signals of applying the ASRE and SNAC techniques along with the no control solution

VI. CONCLUSION

In this paper, we have analyzed the qualitative behavior and optimal control strategy of an SIR model. Two control functions (ASRE and SNAC) have been used, for vaccinating the susceptible populations and for controlling the treatment efforts to the infected populations. We have also studied and determined the optimal vaccination and treatment to minimize the number of infective and susceptible populations as well as the cost due to vaccination and treatment. Finally, efficiency analysis has been performed to determine that the vaccinating to the susceptible populations is better than treatment control to infected populations in order to minimize the infected individuals. The entire study of this paper is mainly based on the deterministic framework and our proposed model is valid for large population. The work is a theoretical modelling and it can be further justified using experimental results.

BIOGRAPHY

Ahmad J. Abougarair, was born in Libya, in 1975. He received the B.S. degree in Electrical and Computer Engineering, in 1998; the M.S. degree in Control and Computer, in 2006; and the Ph.D. degree in Control Engineering in 2018. He is Associate Professor with the Electrical and Electronics Engineering, University of Tripoli. He has published more than 50 papers in local and international conferences and journals and he is an editor and reviewer for many international journals in the field of control and automation. He is an international program committee member, steering committee member and scientific committee member in several international conferences at different countries. His research interests include intelligent control, autonomous vehicles, and the applications of soft computing in modeling and control. http://orcid.org/0000-0001-9738-4888.

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