Stability Analysis of Controlled DC Motor

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Abstract—**Controlled dc motors are nonlinear systems, that show a nonlinear action in their operation including, sub-harmonics and chaos when they work outside their design specifications. This nonlinearity forces the motor changing its normal operation to a random-like behaviour; In this paper, the nonlinear dynamics of DC motors are investigated. It is shown that the concept of the Poincaré map approach and the monodromy matrix method can be successfully applied to determine the stability of DC motors.**

Index Terms: **DC motor, Monodromy matrix, nonlinear behaviour, Poincaré map.**

I. INTRODUCTION

The performance of any system is generally evaluated by its steady-state and dynamic behaviour. When performing a steady-state analysis, the existence and location of periodic solutions are of concern and can be eased by deriving a discrete map that describes the dynamics of the system, and finding its fixed point. When performing a dynamic analysis however, stability and transient response are greatly focused on and can be studied using the closed loop eigenvalues of the system.

In this paper, the stability of the controlled DC motor will be studied using two different approaches. The first is the conventional Poincaré map approach for studying stability of any periodic system, based on:

- Deriving the Poincaré map that describes the dynamics of the system and finding its fixed points.
- Linearizing the map around the fixed points and finding the eigenvalues of the jacobian of this map.
- If all eigenvalues have a magnitude less than unity, the system is stable, otherwise the system is unstable and nonlinear behaviour may exist.

The essence of this method lies in the capture of the dynamics in the small neighbourhood of a periodic orbit. One drawback of using this method in electrical circuits is that sometimes it is difficult to derive the Poincaré map of the system analytically because the equations of the system are transcendental. Therefore this map can only be calculated numerically [1-4].

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The second method is based on deriving the monodromy matrix of the system, which is the fundamental solution matrix of the system for one complete cycle, and finding the Floquet multipliers. The Floquet multipliers are the eigenvalues of the monodromy matrix.

This paper is constructed as follows: first a general DC motor driven by a chopper circuit is modeled using a sampled data model. Based on this map, the existence and location of a periodic orbit are obtained. Next the nonlinear phenomena in the system including bifurcation and chaos are shown by simulation. Finally the stability analysis of the system is considered using the two methods mentioned above.

II. TOPOLOGY AND OPERATION OF CONTROLLED DC MOTOR

A simplified block diagram of a speed controller DC motor is shown in Figure 1. It consists of a feedback loop which observes the speed variation and adjusts the duty cycle *d*. The switch *S* is controlled by a comparator which compares a control signal V_{con} with a periodic saw-tooth waveform V_{ramp}. Switch *S* is open when $V_{con} > V_{ramp}$ and is closed when $V_{con} < V_{ramp}$ as shown in Figure 2.

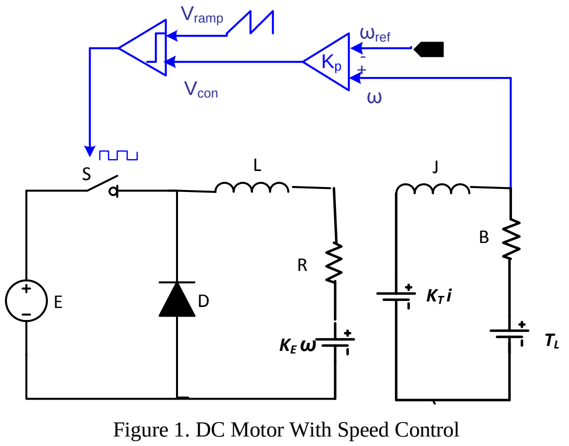


Figure 1. DC Motor With Speed Control

The control signal V_{con} is derived from the speed signal through a standard error amplifier. Using a simple proportional feedback controller, the control signal can be written as:

$$V_{con}(t) = K_p(\omega(t) - \omega_{ref}) \quad (1)$$

where ω_{ref} is the reference speed, the desired speed, and K_p is the gain of the feedback amplifier. The ramp signal can be expressed as:

$$V_{ramp}(t) = V_L + \Delta V\left(\frac{t}{T} \bmod 1\right) \quad (2)$$

where $\Delta V = (V_U - V_L)$, V_L, V_U are the lower and upper voltages of the ramp signal respectively and T is the period of one cycle.

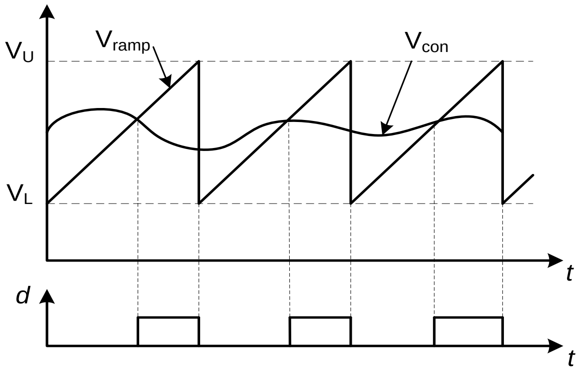


Figure 2. Typical Operation Waveforms of dc Motor with Speed Mode Control

The system is governed by two sets of linear differential equations related to the ON and OFF states of the controlled switch. The inductor current i_L and the speed of the motor ω are taken as state variables.

The equations that represent the dynamics of the system in state space form are:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) = \begin{cases} \mathbf{f}_+(\mathbf{x}(t), t) & S: \text{ON} \\ \mathbf{f}_-(\mathbf{x}(t), t) & S: \text{OFF} \end{cases} \quad (3)$$

where $\mathbf{f}_+(\mathbf{x}(t), t)$, $\mathbf{f}_-(\mathbf{x}(t), t)$ are the two smooth vector fields when the switch is ON and OFF respectively, defined as:

$$f_+(x(t), t) = A_{ON}\, x(t) + B_{ON}\, U$$
$$f_-(x(t), t) = A_{OFF}\, x(t) + B_{OFF}\, U \quad (4)$$

$\mathbf{x} = [\omega \quad i_L]^T = [x_1 \quad x_2]^T$ is the state vector, and $U = [T_L \quad E]^T$ is the input vector. **A** and **B** are the system matrices that contain the system parameters, defined as:

$$A_{ON} = A_{OFF} = \begin{bmatrix} -B/J & K_t/J \\ -K_e/L & -R/L \end{bmatrix}$$

$$B_{ON} = \begin{bmatrix} -1/J & 0 \\ 0 & -1/L \end{bmatrix} \text{ and } B_{OFF} = \begin{bmatrix} -1/J & 0 \\ 0 & 0 \end{bmatrix} \quad (5)$$

The switching instant occurs when the control signal is equal to the ramp signal; thus the switching condition $h(\mathbf{x}, t)$ is defined through feedback proportional control as:

$$h(x, t) = K_p(\omega(t) - \omega_{ref}) - V_{ramp} = 0 \quad (6)$$

In general, the circuit gives an average speed close to the desired value with a periodic ripple equal to the period of the driving clock as shown in Figure 3. The output speed shows a repetitive oscillation with a fixed speed ripple, also called period-1 operation. However, nonlinear phenomena such as bifurcation and chaos will appear when one of the circuit parameters is varied, the circuit parameters is chosen as L=53.7e-3 H; R=2.8 ohm ; Kp=0.7; B=0.000275; J=0.000557; Tl=0.38; ωref=100 rad/sec ; Ke=0.1356;Kt=0.1324; [1-5].

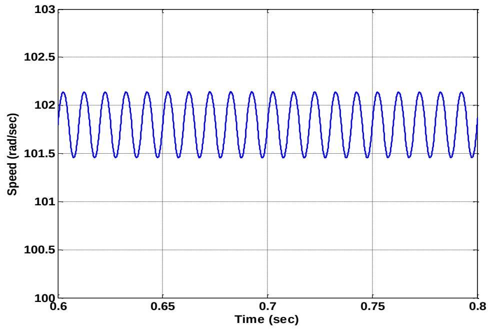


Figure 3. Simulation Results, Period -1 Operation

III. NONLINEAR PHENOMENA IN DC MOTOR

DC motors driven by a chopper circuit are nonlinear dynamic systems. The nonlinearities arise mainly due to switching power devices in control circuits, and nonlinearities in passive components such as inductors and capacitors [6]. Controlled DC motors exhibit various types of complex behaviour such as bifurcations and chaotic operation. These phenomena are called the nonlinear dynamics of the system [5-8]. In this section, nonlinear phenomena in a controlled DC motor are studied using the time waveforms of state variables, phase portraits and bifurcation diagrams. The input voltage is used as bifurcation parameters to investigate the changing behaviour of the system. Results are validated theoretically, showing good agreement with simulation.

A. *Simulations results*

In order to study the dynamics of the controlled system, Equation (3), which describes the dynamics of DC motors, is solved using MATLAB/ SIMULINK. The switching instants are determined by comparing the ramp signal with the control signal. The input voltage was used as the bifurcation parameter and was varied from 50V to 80V. The states were sampled at the start of each cycle of the ramp; thus a sampled data map was obtained. The sampled values of the speed (neglecting the initial transient) are plotted against the bifurcation parameter to obtain the bifurcation diagram, shown in Figure 4.

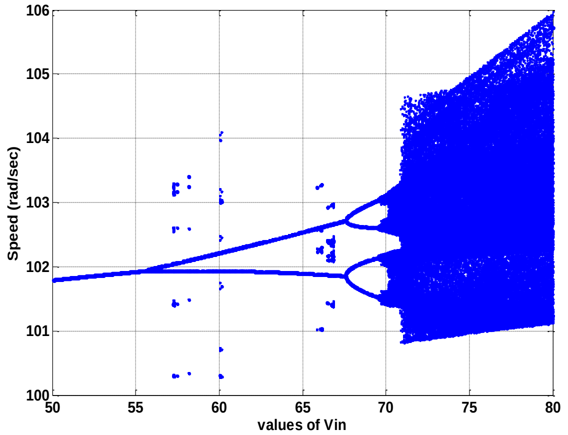
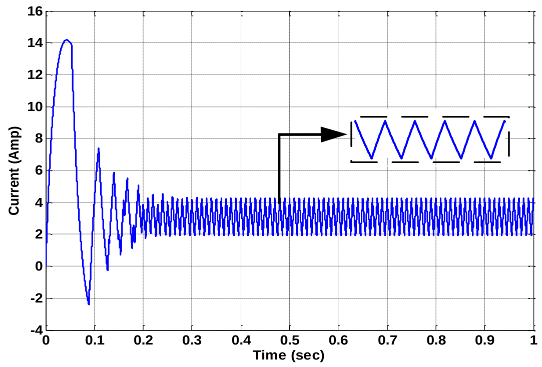


Figure 4. Bifurcation Diagram of the System as the Input Voltage is Changed

Normally, the controlled DC motor is designed to operate in period-1. This is when the input voltage is less than 55.6V for this specific system. However, increasing the input voltage, a period-doubling bifurcation occurs at 55.6V and the stability of the period-1 is lost to another periodic orbit, period-2.

This periodic solution continues until the input voltage is near 67V then it loses its stability again and bifurcates to period-4. As the input voltage increases further, a cascade of period- doubling takes place and at some point the system will enter into a chaotic region at an input voltage of around 70V. Above this value the system begins to operate in the chaotic region and exhibits some complex behaviour.

The speed and the inductor current waveforms in the time domain and the state space for an input voltage of 50V are shown in Figures 5 and 6, respectively. The results indicate that the system is working in period-1 operation at this operating point.



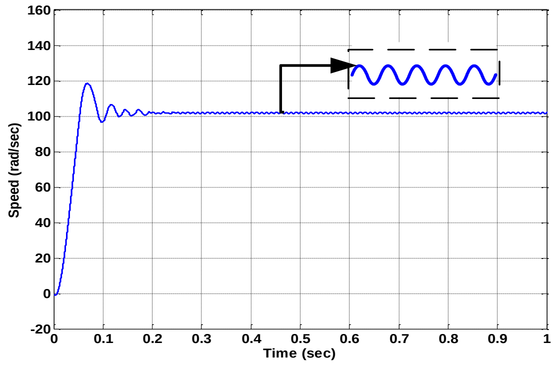


Figure 5. Period-1 Waveforms of the DC Motor, $E = 50$V

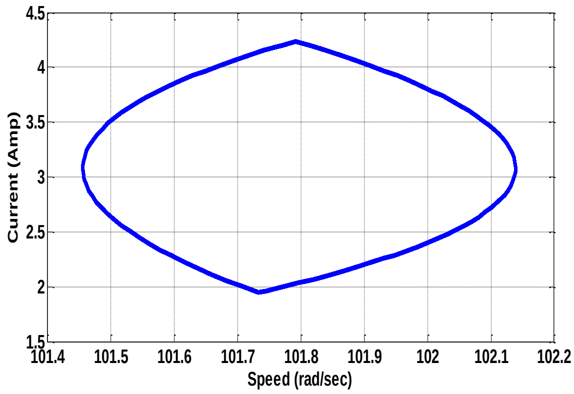


Figure 6. Phase Portrait of the System, $E = 50$V

System waveforms as the input voltage increases to 60V are shown in Figures 7 and 8. It is obvious that the system is working in period-2 operation i.e. the states repeat themselves every two switching cycles.

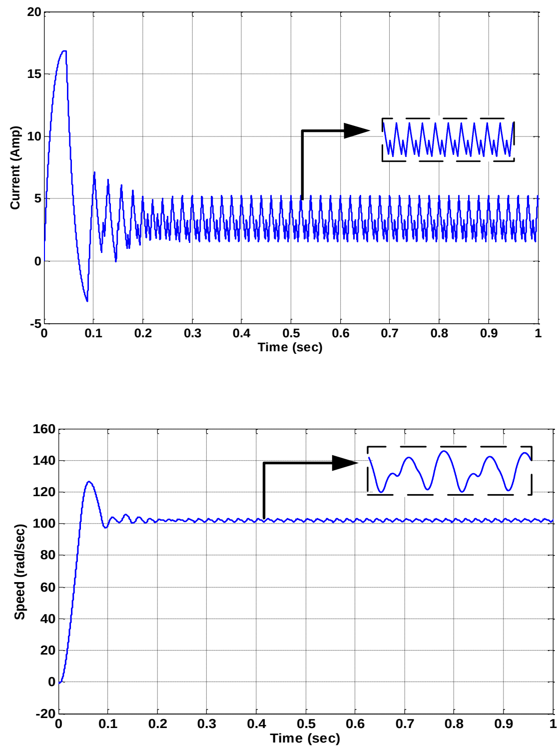


Figure 7. Period-2 waveforms of the DC motor, $E = 60$V

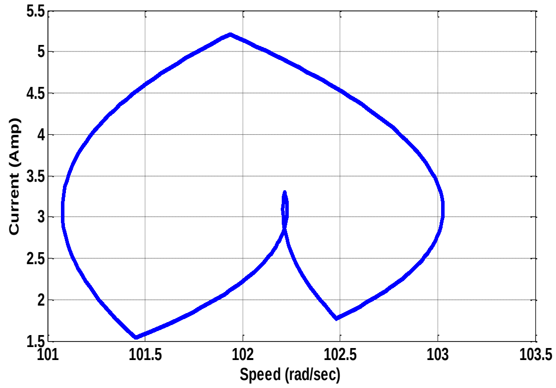


Figure 8. Phase portrait of the system, E = 60V

As the input voltage increases further, to 70V, the system operates in a chaotic state as shown in Figure 9. The phase portrait at an input voltage of 70V shows a bounded solution with non-periodic motion (Figure 10).

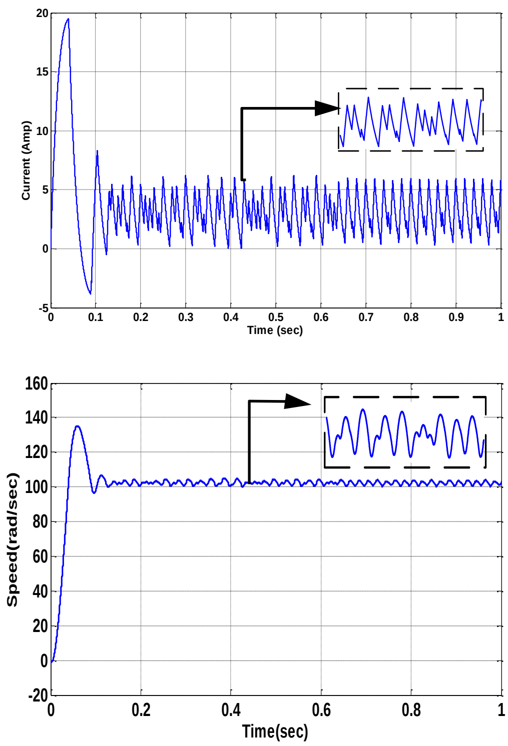


Figure 9. Chaotic waveforms of the DC motor, E = 70V

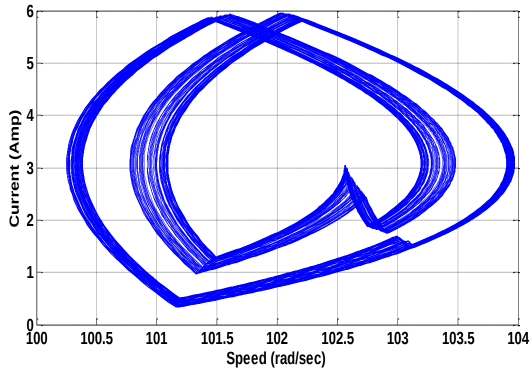


Figure 10. Phase portrait of the system, E = 70V

B. Discrete maps and periodic solutions

In switching systems, the steady-state operation is a periodic orbit not an equilibrium point. Furthermore, this periodic orbit is non-smooth due to the switching action. One way to check the existence of the periodic orbit and find its location analytically is to derive a discrete map that describes the system [9-12]. In this section, the data sampled model in the form of a stroboscopic map is derived, where the state variables are sampled at the beginning of each cycle, to get a discrete model. Assuming the controlled DC motor operates in the nominal period-1 steady-state, in which there is only one switching in one clock cycle occurring at the time instant $d'T$ as shown in Figure 11.

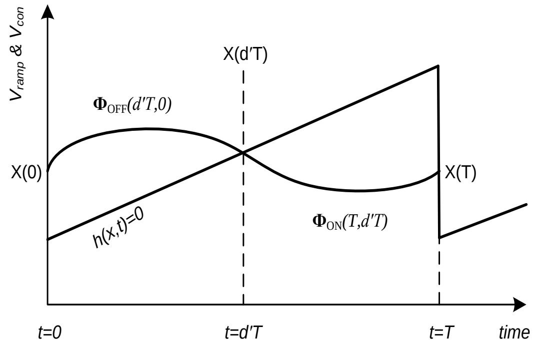


Figure 11. Typical Periodic Solution of the Controlled DC Motor

Since the controlled DC motor is a smooth piecewise linear system, the dynamics of the system before and after the switching can be described by a linear time invariant (LTI) ordinary differential equation 3. Therefore the solution of the system for each interval can be obtained directly by the exponential matrix method.

During the first interval, the ramp is crossed at $t=d'T$, the switch is OFF, and the solution of the system will be:

$$\mathbf{x}(d'T)=e^{\mathbf{A}_{\mathrm{OFF}}(d'T)}\,\mathbf{x}(0)+\int_0^{d'T} e^{\mathbf{A}_{\mathrm{OFF}}(d'T-\tau)}\,\mathbf{B}_{\mathrm{OFF}}\,U\,d\tau$$

$$=\mathbf{\Phi}_{\mathrm{OFF}}(d'T,0)\mathbf{x}(0)+\mathbf{\Gamma}_{\mathrm{OFF}}(d') \quad (7)$$

where $\mathbf{\Phi}_{\mathrm{OFF}}(d'T,0)$, is the state transition matrix during the first interval.

Likewise, during the second interval, the switch is ON and the state vector is given by:

$$\mathbf{x}(T) = e^{\mathbf{A}_{ON}(T-d'T)}\,\mathbf{x}(d'T) + \int_{d'T}^{T} e^{\mathbf{A}_{ON}(T-\tau)}\mathbf{B}_{ON}\,U\,\mathrm{d}\tau \quad (8)$$

$$= \mathbf{\Phi}_{ON}(T, d'T)\mathbf{x}(d'T) + \mathbf{\Gamma}_{ON}(d')$$

where $\mathbf{\Phi}_{ON}(T, d'T)$, is the state transition matrix when the switch is ON.

Since the vector field of the system is piecewise linear (the solution of the system is continuous everywhere but only piecewise differentiable), the final state just before a switching instant can be taken as the initial state after the switching. This yields:

$$\mathbf{x}(T) = \mathbf{\Phi}_{ON}(T, d'T)\{\mathbf{\Phi}_{OFF}(d'T,0)\mathbf{x}(0) + \mathbf{\Gamma}_{OFF}(d')\} + \mathbf{\Gamma}_{ON}(d') \quad (9)$$

This is a sampled data map of the system, also known as the Poincaré map, and can be simplified to:

$$\mathbf{x}(T) = \mathbf{\Phi}(T,0)\,\mathbf{x}(0) + \mathbf{\Gamma}(d') \quad (10)$$

where

$\mathbf{\Phi}(T,0) = \mathbf{\Phi}_{ON}(T, d'T) \times \mathbf{\Phi}_{OFF}(d'T,0)$ and

$\mathbf{\Gamma}(d') = \mathbf{\Phi}_{ON}(T, d'T)\,\mathbf{\Gamma}_{OFF}(d') + \mathbf{\Gamma}_{ON}(d')$

For a periodic solution $\mathbf{x}(T) = \mathbf{x}(0)$ must be satisfied thus the following expression for $\mathbf{x}(0)$ can be obtained:

$$\mathbf{x}(0) = [\mathbf{I} - \mathbf{\Phi}(T,0)]^{-1} \times \mathbf{\Gamma}(d') \quad (11)$$

where $\mathbf{I}$ is the identity matrix of the same order as the system matrix $\mathbf{A}_{ON}$, and $\mathbf{x}(0)$ is the fixed point of the map. Since the nonlinear equation (11) is a function of d', the equation can be solved with the switching equation $h(\mathbf{x}(0), d') = 0$ to obtain the duty cycle.

The switching equation $h(\mathbf{x}(0), d') = 0$ at the switching instant is given by:

$$h(\mathbf{x}(0), d') = [1 \quad 0][\mathbf{\Phi}_{OFF}(d'T,0)\mathbf{x}(0) + \mathbf{\Gamma}_{OFF}(d')] - \omega_{ref} - \frac{V_L + \Delta V\, d'}{K_P} \quad (12)$$

Substituting equation (11) into equation (12), a transcendental equation will be obtained which involves only one unknown, the switching instant:

$$[1 \quad 0]\left[\mathbf{\Phi}_{OFF}(d'T,0)[\mathbf{I} - \mathbf{\Phi}(T,0)]^{-1} \times \mathbf{\Gamma}(d') + \mathbf{\Gamma}_{OFF}(d')\right] - \omega_{ref} - \frac{V_L + \Delta V d'}{K_P} = 0 \quad (13)$$

This transcendental nonlinear equation can be solved numerically with a method such as the *Newton-Raphson* method to obtain the duty ratio. Figure 12 shows the numerical values obtained for the duty cycle for different values of the input voltage.

Once the duty cycle is calculated, the fixed point of the sampled data map $\mathbf{x}(0)$ can be obtained using equation (11) which corresponds to the location of the periodic orbit of the continuous system.

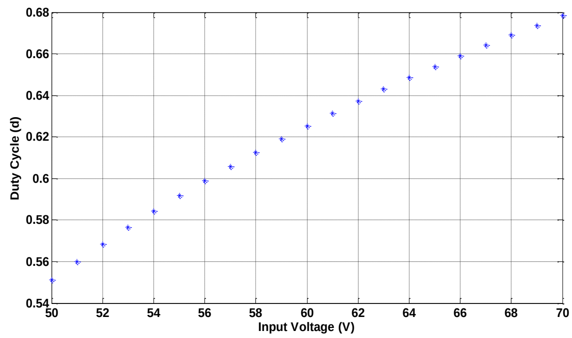


Figure 12. Evaluation of the Duty-Cycle for Period-1 Operation

IV. STABILITY ANALYSIS OF THE DC MOTOR USING POINCARE MAP METHOD

The stability of any periodic system exhibiting nonlinear behaviour such as the controlled DC motor is usually carried out as follows:

- Deriving the discrete time map (Poincaré map) of the system.
- Calculating the jacobian matrix of this map by linearizing around its fixed point.
- Finally, finding the eigenvalues of the jacobian. The natural response of the system will decay to zero (the system will be stable) if and only if the eigenvalues of the jacobian have a magnitude less than unity, otherwise the system is unstable.

In a controlled DC motor this map can be established in a number of ways; one way is by sampling the state variables at the beginning of each switching period T as described in the previous section.

A. Derivation of jacobian matrix of the DC motor

In section 3.b the Poincaré map of the DC motor has been derived (equation 10). For simplicity, this can be written as:

$$\mathbf{x}(T) = f(\mathbf{x}(0), d') \quad (14)$$

In order to check the stability of the system, we first need to linearize this map around its fixed point $\mathbf{x}(0)$ to obtain the jacobian matrix $\frac{\partial \mathbf{x}(T)}{\partial \mathbf{x}(0)}$. This can be achieved by differentiating equation (14) with respect to $\mathbf{x}(0)$ and using the series rule:

$$\frac{\partial \mathbf{x}(T)}{\partial \mathbf{x}(0)} = \frac{\partial f(\mathbf{x}(0), d')}{\partial \mathbf{x}(0)} + \frac{\partial f(\mathbf{x}(0), d')}{\partial d'} \times \frac{\partial d'}{\partial \mathbf{x}(0)} \quad (15)$$

To calculate $\frac{\partial d'}{\partial \mathbf{x}(0)}$, differentiation of the switching manifold $h(\mathbf{x}(0), d')$ with respect to $\mathbf{x}(0)$ is needed; this yields:

$$0 = \frac{\partial h(\mathbf{x}(0), d')}{\partial \mathbf{x}(0)} + \frac{\partial h(\mathbf{x}(0), d')}{\partial d'} \times \frac{\partial d'}{\partial \mathbf{x}(0)} \quad (16)$$

By substituting equation (16) into (15), the following expression of the jacobian matrix is obtained:

$$\frac{\partial \mathbf{x}(T)}{\partial \mathbf{x}(0)} = \left(\frac{\partial f(\mathbf{x}(0), d')}{\partial \mathbf{x}(0)}\right) - \left(\frac{\partial f(\mathbf{x}(0), d')}{\partial d'}\right) \times \left(\frac{\partial h(\mathbf{x}(0), d')}{\partial d'}\right)^{-1} \times \left(\frac{\partial h(\mathbf{x}(0), d')}{\partial \mathbf{x}(0)}\right) \quad (17)$$

This is a general form of the jacobian matrix for any switching systems. By differentiating the Poincaré map of the controlled DC motor equation (10) and (12) with respect to $\mathbf{x}(0)$ and d', the final expression for the jacobian of the voltage controlled DC motor is obtained.

$$\frac{\partial \mathbf{x}(T)}{\partial \mathbf{x}(0)} = \mathbf{\Phi}(T,0) - \frac{-\mathbf{\Phi}_{\mathrm{ON}}(\mathbf{B}_{\mathrm{ON}} - \mathbf{B}_{\mathrm{OFF}})U([1 \quad 0]\mathbf{\Phi}_{\mathrm{OFF}})}{[1 \quad 0](\mathbf{A}_{\mathrm{OFF}}\mathbf{x}(d'T) + \mathbf{B}_{\mathrm{OFF}}U) - \frac{\Delta V}{K_P T}} \quad (18)$$

B. Calculating the eigenvalues of the jacobian matrix

In order to calculate the jacobian matrix and hence to check the stability of the system, one needs to find the location of the periodic orbit and the switching instant. This can be achieved numerically by solving the discrete map of the system with the switching equation to generate a nonlinear equation (13) whose roots will define the switching instant. Once the switching instant has been identified, utilizing the fact that the system is LTI before and after the switching, it is possible to locate the limit cycle. Once these values are found, the jacobian matrix (18) can be expressed as a function of the input voltage and its eigenvalues can be calculated. Figure 13 shows the evaluation of the eigenvalues of the jacobian matrix for different values of the input voltage, clearly indicating the loss of stability through a smooth period-doubling bifurcation around an input voltage of 56V as demonstrated by the previous simulation and experimental results.

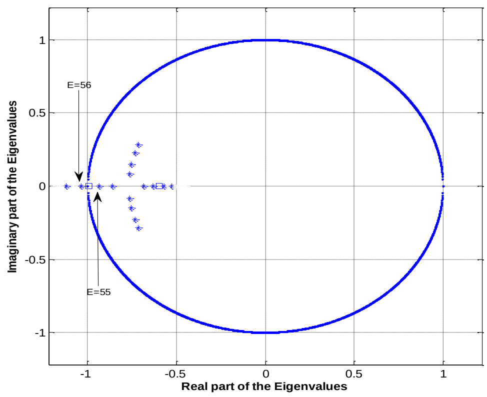


Figure 13. Evaluation of the Eigenvalues of the Iacobian Matrix as the Input Voltage Changes from 50V to 60V

C. Stability Analysis of the controlled DC motor Using the Monodromy Matrix Method

In this section an alternative approach for determining the stability of any switching systems is proposed based on linearizing the system around the whole periodic orbit. The new method requires the calculation of the monodromy matrix of the controlled DC motor. The monodromy matrix of any switching system is the product of the state transition matrices before, during and after the switching for one complete cycle.

D. Derivation of the monodromy matrix of the DC motor

The controlled DC motor is a nonlinear system that represents different circuit topologies within one switching cycle. For the continuous conduction mode, there are two topologies. In each topology, the system can be described by linear state equations. However, for a complete switching cycle, the system becomes piecewise linear and the solution is not defined at the switching instant. The solution of such a system can be defined on the basis of differential inclusions using Filippov's concept [13-15].

To define the solution of a system while it is on the switching manifold, Filippov suggested that the vector field at the switching instant will not be a single valued function but a set valued function whose limits are the values of the vector fields before and after the switching. As a result the original piecewise equation (4) that describes the dynamics of the DC motor has to be extended into a differential inclusion **F** (**x**, t) as:

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(\mathbf{x}, t) = \begin{cases} \mathbf{f}_{-}(\mathbf{x}, t) & \mathbf{x} \in V_{-} \\ \overline{co}\{\mathbf{f}_{-}(\mathbf{x}, t), \mathbf{f}_{+}(\mathbf{x}, t)\} & \mathbf{x} \in \Sigma \\ \mathbf{f}_{+}(\mathbf{x}, t) & \mathbf{x} \in V_{+} \end{cases} \quad (19)$$

where $\mathbf{f}_-(\mathbf{x}(t),t)$ and $\mathbf{f}_+(\mathbf{x}(t),t)$ are the two smooth vector fields before and after the switching. They are defined as:

$$\mathbf{f}_+(\mathbf{x}(t),t)=\mathbf{A}_{ON}\,\mathbf{x}(t)+\mathbf{B}_{ON}\,U=\begin{bmatrix}\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\\[2ex]\dfrac{-K_e\,x_1(t)-R\,x_1(t)+E}{L}\end{bmatrix} \quad (20)$$

$$\mathbf{f}_-(\mathbf{x}(t),t)=\mathbf{A}_{OFF}\,\mathbf{x}(t)+\mathbf{B}_{OFF}\,U=\begin{bmatrix}\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\\[2ex]\dfrac{-K_e\,x_1(t)-R\,x_1(t)}{L}\end{bmatrix} \quad (21)$$

It is obvious from equations (20) and (21) that there is a discontinuity when the main switching element passes from ON state to OFF state, since $\mathbf{f}_-(\mathbf{x}(t),t)\neq\mathbf{f}_+(\mathbf{x}(t),t)$.

The two dimensional state space is now divided into three parts V_-, V_+ and Σ as shown in Figure 14 where V_- is the time interval during which the switch is OFF, V_+ is the time interval during which the switch is ON and Σ is the switching instant.

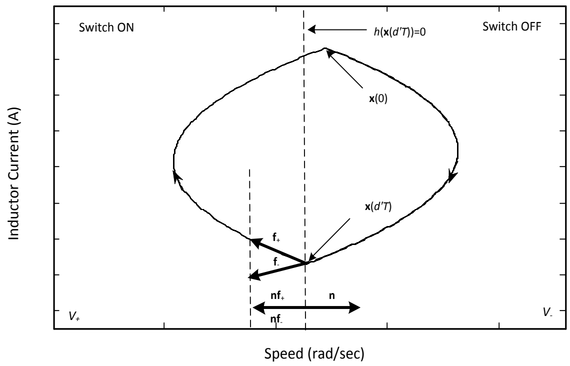


Figure 14. Transversal Intersections in the Orbit of the Controlled DC Motor

The smallest closed convex set is defined as:

$$\overline{co}\{\mathbf{f}_-,\mathbf{f}_+\}=\begin{bmatrix}\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\\[2ex](1-q)\dfrac{-K_e\,x_1(t)-R\,x_1(t)+E}{L}-\\ q\dfrac{-K_e\,x_1(t)-R\,x_1(t)}{L}\end{bmatrix}$$

$$\forall q\in[0,1] \quad (22)$$

The normal to the switching manifold **n** is given by:

$$\mathbf{n}=\nabla\left(h\,\mathbf{x}(t_\Sigma),t_\Sigma\right) \quad (23)$$

$$=\begin{bmatrix}\dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2}\end{bmatrix}^T=\begin{bmatrix}1\\0\end{bmatrix}$$

Therefore the projections of $\mathbf{f}_-$ and $\mathbf{f}_+$ on the hyper-surface Σ are given by:

$$\mathbf{n}^T\mathbf{f}_-=[1\ \ 0]\times\begin{bmatrix}\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\\[2ex]\dfrac{-K_e\,x_1(t)-R\,x_1(t)}{L}\end{bmatrix}=\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J} \quad (24)$$

$$\mathbf{n}^T\mathbf{f}_+=[1\ \ 0]\times\begin{bmatrix}\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\\[2ex]\dfrac{-K_e\,x_1(t)-R\,x_1(t)+E}{L}\end{bmatrix}=\dfrac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J} \quad (25)$$

The extension of a discontinuous system (4) into a convex differential inclusion (14) is known as Filippov's convex method. The solution is unique for every initial condition, if it crosses the hyper-surface transversally and spends almost zero time on the switching manifold.

A necessary condition for a transversal intersection at Σ is [15]:

$$\mathbf{n}^T\mathbf{f}_-(\mathbf{x},t)\times\mathbf{n}^T\mathbf{f}_+(\mathbf{x},t)\rangle\,0 \qquad \mathbf{x}(t)\in\Sigma \quad (26)$$

$$\frac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\times\frac{K_t\,x_2(t)-B\,x_1(t)-T_L}{J}\rangle\,0$$

Therefore the solution is unique as shown in Figure 15.

The period-1 limit cycle of the system, given in Figure 15, shows that the trajectory crosses the switching manifold twice, at $d'T$ and T. Therefore the fundamental solution matrix for one complete cycle, *the monodromy matrix*, is given by:

$$\mathbf{M}(T,0)=\mathbf{S}_2(T)\times\mathbf{\Phi}_{ON}(T,d'T)\times\mathbf{S}_1(d'T)\times\mathbf{\Phi}_{OFF}(d'T,0) \quad (27)$$

where $\mathbf{\Phi}_{ON}(T,dT)$, $\mathbf{\Phi}_{OFF}(dT,0)$ are the state transition matrices during the ON and OFF intervals, respectively, and they are calculated by the exponential matrix.

$\mathbf{S}_1$ and $\mathbf{S}_2$ are the state transition matrices during switching, also called the *saltation matrices*, and they are calculated by the following formula [1, 13-15]:

$$\mathbf{S} = \mathbf{I} + \frac{\left(\mathbf{f}_+\left(\mathbf{x}(t_\Sigma), t_\Sigma\right) - \mathbf{f}_-\left(\mathbf{x}(t_\Sigma), t_\Sigma\right)\right)\mathbf{n}^T}{\mathbf{n}^T \mathbf{f}_-\left(\mathbf{x}(t_\Sigma), t_\Sigma\right) + \dfrac{\partial h\left(\mathbf{x}(t_\Sigma), t_\Sigma\right)}{\partial t}} \tag{28}$$

where t_Σ is the switching time (the time at which the solution hits the switching manifold).

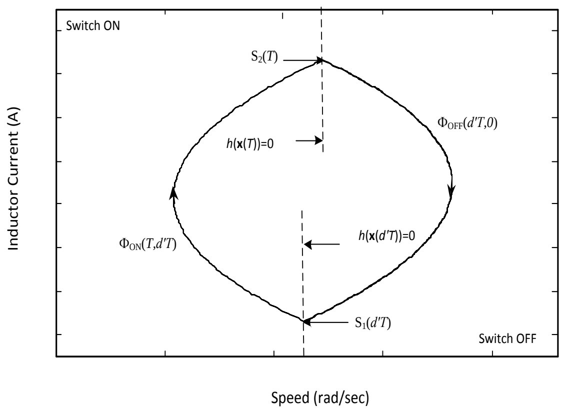


Figure 15. Period-1 Limit Cycle of the DC Motor

The switching manifold is defined by a scalar indicator function $h(\mathbf{x},t)=0$, thus the derivative of $h(\mathbf{x},t)$ with respect to t for period-1 operation $t\in(0,T)$ is:

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\frac{\Delta V}{K_p T} \tag{29}$$

Substituting equations (23) – (29) into equation (28) at the switching instant $t = d'T$, the saltation matrix $\mathbf{S}_1$ can be calculated:

$$\mathbf{S}_1 = \mathbf{I} + \begin{bmatrix} 0 & 0 \\ \dfrac{E/L}{\dfrac{K_t x_2(t) - B x_1(t) - T_L}{J} - \dfrac{\Delta V}{K_p T}} & 0 \end{bmatrix} \tag{30}$$

To calculate $\mathbf{S}_2$, the time derivative of the switching manifold at $t = T$ is needed. Since the switching manifold is discontinuous (with respect to time) at this point the time derivative will be infinite. Therefore the saltation matrix at this point is the identity matrix.

Knowing $\mathbf{S}_1$ and $\mathbf{S}_2$, it is possible to calculate the eigenvalues of the monodromy matrix which, for a period-1 response, must have amplitudes less than 1.The total fundamental solution matrix over one complete cycle of the buck converter is:

$$\mathbf{M}(T,0) = e^{\mathbf{A}_{ON}\, dT} \times \left(\mathbf{I} + \begin{bmatrix} 0 & 0 \\ \dfrac{E/L}{\dfrac{K_t x_2(t) - B x_1(t) - T_L}{J} - \dfrac{\Delta V}{K_p T}} & 0 \end{bmatrix}\right) \times e^{\mathbf{A}_{OFF}\, d'T} \tag{31}$$

The stability of the periodic orbit can be determined by obtaining the Floquet multipliers which are the eigenvalues of the monodromy matrix.

E. Calculating the Floquet multipliers

The stability of the system can be determined by calculating the Floquet multipliers which are the eigenvalues of the fundamental solution matrix $\mathbf{M}(T,0)$. Common problems that have to be addressed here are the location of the limit cycle and the times at which the switching take place. This can be achieved numerically by deriving a nonlinear function equation 13 whose root will define the switching instant. Once the switching instants have been identified, utilizing the fact that the system is LTI before and after the switching, it is possible to locate the limit cycle. Once these values are found, the monodromy matrix can be expressed as a function of the input voltage using equation 31. The computed loci of the eigenvalues with varying input voltage are shown in Figure 16. The figure shows that the system loses its stability through a smooth period-doubling bifurcation at an input voltage of around 55.6V. This result is in perfect agreement with the previous analytical and simulation results.

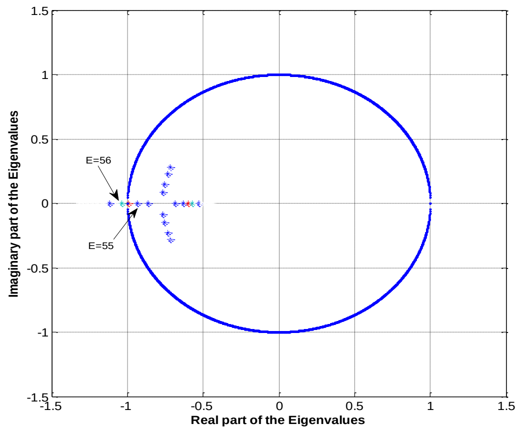


Figure 16. Loci of the Eigenvalues of the Monodromy Matrix for Different Input Voltages

V. CONCLUSION

This paper focused on the stability analysis of the periodic orbit of the DC motor with speed mode control operating in continuous conduction mode. The existence and location of the periodic solution of the system has been proven analytically in this paper. It was shown that the system exhibits nonlinear phenomena, including bifurcation and chaos as the input voltage or the control parameter is changed. The nonlinearities have been shown analytically, by simulation validated. The stability analysis of the speed controlled DC motor has been considered using two approaches; the first is the Poincaré map method which is the conventional method for studying the stability of any periodic orbits. The second approach is based on deriving the state transition matrices before, during and after the switching takes place for one switching cycle (the monodromy matrix). Unsurprisingly, both methods give the same results; however the stability analysis using the monodromy matrix method was easier to implement when compared with the Poincaré map method. This represents a first step towards developing a technique for controlling bifurcations in controlled DC motors which will be used in future work to stabilize the nonlinear behaviour in these systems

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