Dynamic of Distributed Parameter Systems; From Infinite to Finite Systems

Ahmed Abdulhamid Baaiu
Misurata University, Electrical and Electronic Engineering Department, Misurata, Libya

Ismail Mohammed Albatrookh
Misurata University, Electrical and Electronic Engineering Department, Misurata, Libya
ahmed.baaiu@eng.misuratau.edu.ly

Abstract - **This work focuses on describing the dynamic behavior of distributed parameter systems by means of spatial discretization. In control and system theory literature, distributed parameter systems are simply the systems that are represented mathematically by partial differential equations. Thus, the key notion of this work is representing the dynamic behavior of distributed parameter systems by means of spatial discretization of the dynamics of the system. Spatial discretization, or lumping, aims to transform a partial differential equation to a set of ordinary differential equations. The proposed concept approximates the dynamic behavior of the physical system by lumping it to small tranches and then using the finite differences method to interconnect formally these lumps and simulates the dynamic of the overall system.**

Index Terms - **Dynamic of distributed parameter systems, lumping and discretization, simulation, finite differences.**

I. INTRODUCTION

The description of the dynamic behavior of many problems that appears in engineering, physics, chemistry, electricity or industrial process generally leads to mathematical models contains systems of Partial Differential Equations (PDEs) [1, 2, 3, 4, 5]. In control and system theory literature, such systems are called distributed parameter systems or infinite dimensional systems, contrasted to localized parameter systems or finite dimensional systems which generally modeled by ordinary differential equations. Therefore, mathematical modeling, analysis, simulation and control of distributed parameter systems has an accordingly many traditions and studies varies from theoretical to applied, numerical, methods [1, 3, 5, 6].

The purpose of this work is to practically represent and approximate the dynamic behavior of distributed parameter systems by means of lumping the system and representing it as small parts, see also [6, 7]. Considering each lump as a localized parameter system and then expressing the relations and interconnections between all these parts will lead to a system of ordinary differential equations.

Received 16 February 2015; revised 18 February 2016; accepted 9 March 2016.
Available online 10 March 2016.

Solving the lumped system will approximates and gives the solution of the original system's model; partial differential equations (PDEs). Transforming the system from infinite dimensional to finite dimensional will enable us to resolve the system using conventional numerical methods and simplifies the system analysis, simulation and control.

A particular case of distributed parameter systems is the gas transportation process which generally led by two physical phenomena; convention and diffusion. The manipulation of the transportation process in one dimensional spatial domain is taken as an example in this work and also simulation results are presented, see also [2, 3, 5].

In brief, presenting a theoretical and practical results about modeling and simulation of distributed parameter systems is the major aim of this work.

A. *Distributed parameter systems*

The term distributed parameter systems comes from the fact that the state variables of the modeled system are distributed and defined overall a spatial domain Ω, of the space R^n (for $n = 1,2,3$), in which occurs the phenomena modeled by partial differential equations. Consequently, all the state variables of the system are functions of time (*t*) and space (Ω). As we mentioned earlier, the description of the dynamic behavior of many problems that appears in engineering applications, physics, chemistry, electricity or industrial process generally leads to mathematical models contains systems of Partial Differential Equations (PDEs). Therefore, studying these models is of major interest of so many engineering applications.

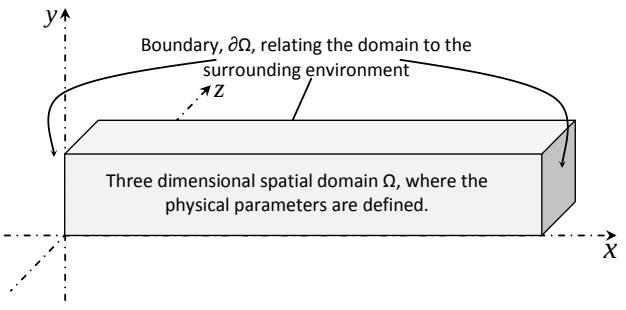

Figure 1. A spatial domain Ω, with its boundary ∂Ω granting the connection with the external environment.

Considering the spatial domain Ω shown in Figure. 1, the boundaries of the spatial domain Ω are mentioned by ∂Ω. The variables of the modeled phenomena are defined on Ω and it can be connected with external sources or other physical phenomena through the boundaries ∂Ω. In brief, the system can interact with its surrounding environment through its boundaries ∂Ω and normally the conditions of the variables at the boundaries ∂Ω will have an effect on the dynamic behavior of the variables in the domain Ω.

B. Mass transfer phenomena

One of the most well-known distributed parameter phenomenon described by partial differential equations is the mass transfer phenomenon [2, 8]. Considering a gas mixture of *k* species flows in the spatial domain Ω, the mathematical model describing the dynamics of this system on overall the spatial domain Ω is given for each spices as [2]:

$$\frac{\partial C_i}{\partial t} = \ div(F_i) \tag{1}$$

where C_i is the molar concentration of the species i and F_i is the molar flow of the species i [2]. In such processes, the molar flow is generally driven by two phenomena; it is composed of a convective flow F_{Ci} and diffusive flow F_{Di}. Hence, the molar flow of species i can be represented as :

$$F_i = F_{Ci} + F_{Di} \tag{2}$$

For simplifying the mathematical manipulations of the model, let us consider a symmetry of the physical variables according to the coordinate axes *y* and *z* and consider only the changes related to the coordinate *x*. Therefore, the convective molar flow of the species *i* can be given as :

$$F_{Ci} = -\,v\,C_i \tag{3}$$

where v is the velocity that drive the convective term according to the coordinate *x*. The diffusive molar flow of the species *i* can be given as :

$$F_{Di} = D_i \frac{\partial C_i}{\partial x} \tag{4}$$

where D_i is the diffusion parameter representing the resistance term and characterizing the diffusive term of the flow. Substituting eq. (3) and eq. (4) in eq. (2) yields :

$$F_i = -vC_i + D_i \frac{\partial C_i}{\partial x} \tag{5}$$

and then substituting eq.(5) in eq. (1) yields :

$$\frac{\partial C_i}{\partial t} = -\,v\frac{\partial C_i}{\partial x} + D_i \frac{\partial^2 C_i}{\partial x^2} \tag{6}$$

Recalling the assumption of symmetry according to the coordinates *y* and *z*, and considering the boundary conditions at the two edges of the coordinate *x*, see also [2, 8], the mathematical model of the system can be represented as :

$$\begin{aligned} &\textit{Model} \begin{cases} \dfrac{\partial C_i}{\partial t} = -v\dfrac{\partial C_i}{\partial x} + D_i \dfrac{\partial^2 C_i}{\partial x^2} \end{cases} \\ &\textit{Boundary conditions} \begin{cases} \left(D_i \dfrac{\partial C_i}{\partial x} - vC_i \right)\Big|_{x=0} = C_{in} v_{in} \\ \left(\dfrac{\partial C_i}{\partial x} \right)\Big|_{x=L} = 0 \end{cases} \end{aligned} \tag{7}$$

where $C_{in}\, v_{in} = F_{in}$ is the molar flux of the spices *i* at the input boundary of the system. From the model given in eq. (7), it is clear that the system is described by a set of partial differential equations and its variables are define on the spatial domain Ω reduced to R^1 and with a coordinates axis x, with *x* varies from 0 to L.

II. CONCEPT OF LUMMPING

The main idea behind the lumping is to approximate the spatial domain as a sum of *n* smaller parts, or tranches, see Fig. 2. As a direct consequence, the variables of the distributed parameter system will be approximated by lumped ones defined on each part, or tranche. Variables in each tranche are intended to interact with its equivalent variables in the neighbor tranches. This interaction can take place through the boundaries of the tranche, and it can represent and approximate the dynamic on the specified tranche. Finally, the overall dynamic of the distributed parameter system can be approximated by considering the dynamic of all tranches.

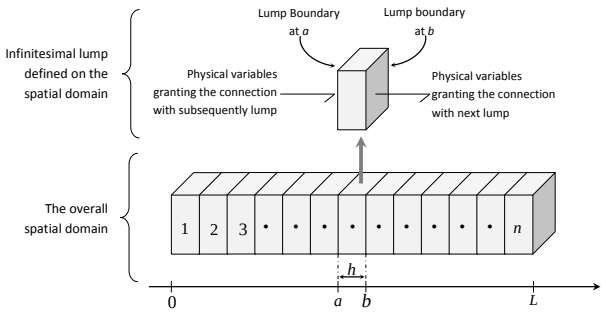

Figure 2. Concept of a lumped distributed parameter system.

As shown in Figure. 2, the spatial domain is lumped to n tranches with width h, consequently $h = (L/n)$. This

lumping approach, Figure. 2, is also often used in system and control engineering practice, where the distributed parameter systems are approximated by lumped ones. Consequently, the control community has made many efforts to extend certain concepts known from finite-dimensional control theory to the distributed parameter case [1, 3, 6, 7].

III. MATHEMATICAL BASIS

According to the mathematical model and its related boundary conditions given in eq. (7), the first step towards a finite dimensional model is to discretize the partial derivatives according to the spatial variable x. This can be achieved using the method of finite differences. The finite difference discretization scheme is one of the simplest forms of discretizing and representing the spatial derivatives. The derivatives of the partial differential equation are approximated by linear combinations of function values at the different parts, or tranches, of the spatial domain. Arbitrary order approximations can be derived from a Taylor series expansion, and according to the model of eq. (7) the first and second derivatives can be given as :

$$\left(\frac{\partial C_i}{\partial x}\right)_j = \frac{(C_i)_j - (C_i)_{j-1}}{h} \tag{8}$$

$$\left(\frac{\partial^2 C_i}{\partial x^2}\right)_j = \frac{(C_i)_{j+1} - 2(C_i)_j + (C_i)_{j-1}}{h^2} \tag{9}$$

where ($j = 1,2,.....,n$) is the tranches index.

According to the expressions of eq. (7), the dynamics related to the convection phenomenon can be expressed as :

$$\dot{C}_C = A_C\, C + B_C\, C_{in} \tag{10}$$

Where the convection matrix A_C is given as:

$$A_C = \left(\frac{-v}{h}\right)\begin{pmatrix} 1 & 0 & 0 & 0 & ..... & 0 & 0 \\ -1 & 1 & 0 & 0 & ..... & 0 & 0 \\ 0 & -1 & 1 & 0 & ..... & 0 & 0 \\ 0 & 0 & -1 & 1 & ..... & 0 & 0 \\ : & : & : & : & & : & : \\ 0 & 0 & 0 & 0 & ..... & 1 & 0 \\ 0 & 0 & 0 & 0 & ..... & -1 & 1 \end{pmatrix} \tag{11}$$

and the input matrix B_C related to the convection phenomenon is given as:

$$B_C = \left(\frac{-v}{h}\right)\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ : \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

In the same way, the dynamics related to the diffusion phenomenon can be expressed as :

$$\dot{C}_D = A_D\, C + B_D\, C_{in} \tag{13}$$

Where the convection matrix A_D is given as:

$$A_D = \left(\frac{D_i}{h^2}\right)\begin{pmatrix} -2 & 1 & 0 & 0 & ..... & 0 & 0 \\ 1 & -2 & 1 & 0 & ..... & 0 & 0 \\ 0 & 1 & -2 & 1 & ..... & 0 & 0 \\ 0 & 0 & 1 & -2 & ..... & 0 & 0 \\ : & : & : & : & & : & : \\ 0 & 0 & 0 & 0 & ..... & -2 & 1 \\ 0 & 0 & 0 & 0 & ..... & 0 & 0 \end{pmatrix} \tag{14}$$

and the input matrix B_D related to the diffusion phenomenon is given as:

$$B_D = \left(\frac{D_i}{h^2}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ : \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

Consequently, and by combining the eq. (10) and eq. (13) to represent the complete dynamics of the system. The set of differential equations representing the finite dimensional model can be given as:

$$\dot{C} = A\, C + B\, C_{in} \tag{16}$$

With $A = A_C + A_D$ and $B = B_C + B_D$. Therefore

$$A = \begin{pmatrix} v/h - (2D_i/h^2) & D/h^2 & 0 & 0 & .. & 0 & 0 & 0 \\ D/h^2 - v/h & v/h - (2D_i/h^2) & D/h^2 & 0 & .. & 0 & 0 & 0 \\ 0 & D/h^2 - v/h & v/h - (2D_i/h^2) & D/h^2 & .. & 0 & 0 & 0 \\ 0 & 0 & D/h^2 - v/h & v/h - (2D_i/h^2) & .. & 0 & 0 & 0 \\ 0 & 0 & 0 & D/h^2 - v/h & .. & 0 & 0 & 0 \\ : & : & : & : & & : & : & : \\ 0 & 0 & 0 & 0 & .. & D/h^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & .. & v/h - (2D_i/h^2) & D/h^2 & 0 \\ 0 & 0 & 0 & 0 & .. & D/h^2 - v/h & v/h - (2D_i/h^2) & D/h^2 \\ 0 & 0 & 0 & 0 & .. & 0 & -v/h & v/h \end{pmatrix} \tag{17}$$

and

$$B = \begin{pmatrix} D/h^2 - v/h \\ 0 \\ 0 \\ 0 \\ : \\ 0 \\ 0 \end{pmatrix} \quad (18)$$

where A is the overall system matrix and B is the overall input matrix.

IV. SIMULATION RESULTS

In order to simulate the distributed parameter system defined in eq. (7) with taking in consideration its related boundary conditions. The spatial domain, on which the system is defined, is discretized in ten equal lumps, $n = 10$ in Fig. 2. The discretized model, infinite dimensional model, will take the form defined by eq. (16) where the system matrix A will be a (10×10) matrix and the input matrix B is (10×1) matrix. Considering the physical parameters given in [8], where $L = 0.5(m)$, $v = 0.02(m/s)$, $D_i = 2.8 \times 10^{-5}(m^2/s)$. Consequently, $h = L/n = 0.05(m)$. According to these physical parameters, the system matrices will be :

$$A = \begin{pmatrix} 0.38 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.39 & 0.38 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.39 & 0.38 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.39 & 0.38 & 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.39 & 0.38 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.39 & 0.38 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.39 & 0.38 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.39 & 0.38 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.39 & 0.38 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0.4 \end{pmatrix} \quad (19)$$

and

$$B = \begin{pmatrix} -0.39 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

The simulated experiment is the response of a transportation column to a steam of a gaseous oxygen; O_2. The process is considered to be initially at equilibrium for a very low concentration of O_2 all along the profile (which means in each tranche in the simulated column). At $(t = 0)$, a step change of a high concentration of O_2 is given at the input of the transportation column. The simulation have been taken for 50 seconds and the results are given in Fig. 3. Fig. 3 shows clearly the concentration profile in the ten parts, tranches, of the spatial domain representing the transportation column. Note that, the first curve represent the concentration in the first tranche, the second for the second tranche and so on.

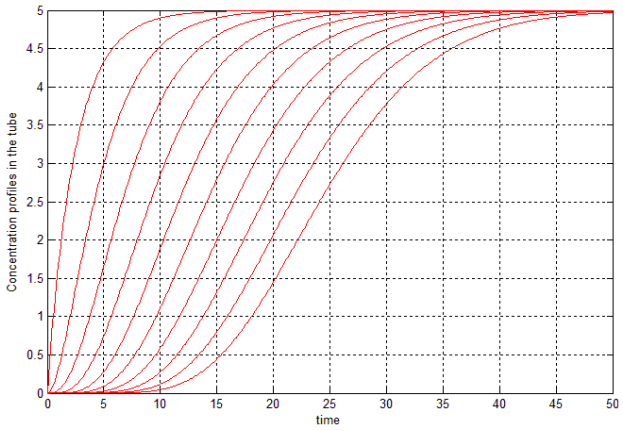

Figure 3. Simulation results.

V. CONCLUSION

This paper discussed the representation of the dynamic behavior of distributed parameter systems and focuses on lumping the spatial domain on which the system is defined to several parts, also called tranches. In order to illustrate the concept, a model of transportation column has been derived from the two main physical phenomena related to the transportation process; the diffusion and the convection. After discretizing and lumping the spatial domain, all the spatial derivatives have been approximated by means of Taylor expansion. The first consequence of the lumping approach is the transformation of the infinite dimensional model to a finite *n*-dimensional model. The lumped model is finally simulated. The methodology presented exhibits some interesting features of transforming the infinite dimensional distributed parameter model to a finite dimensional model, and the Taylor expansion conserves the linear properties of the spatial derivatives.

REFERENCES

[1] A. Baaiu, F. Couenne, L. Lefevre, Y. Le Gorrec, M. Tayakout, "Structure-preserving infinite dimensional model reduction: Application to adsorption processes" *Journal of Process Control*, vol. 19, Issue 3, pp. 394-404, March 2009.

[2] D. Ruthven, *Principles of adsorption and adsorption processes*, 1st Edition. New York, U. S. A. John Wiley and sons, 1994.

[3] V. Duindam, A. Macchelli, S. Stramigioli, H. Bruyninckx, *Modeling and Control of Complex Physical Systems, The Port-Hamiltonian Approach*, 1st Edition, Springer-Verlag Berlin Heidelberg, 2009.

[4] E. Kreyszig, *Advanced Engineering Mathematics*, Third Edition, John-Wiley & sons, Inc, 1992.

[5] D. Coughanowr, *Process systems Analysis and Control*, Second Edition, McGraw-Hill, Inc, 1991.

[6] A. Baaiu, F. Couenne, L. Lefevre, and Y. Le Gorrec, "Energy-preserving method for spatial discretization: application to an adsorption column" *Computer Aided Chemical Engineering*, vol. 25, pp. 727-732, 2008.
[7] A. Baaiu, F. Couenne, L. Lefevre, Y. Legorrec and M. Tayakout, "On Energy based discretization of an adsorption column", in *Proc. ADCHEM (International Symposium on Advanced Control of Chemical Processes)*, Brazil, April 2006, pp. 8-16.
[8] A. Baaiu, "Hamiltonian approach for modeling, estimation and control of a separation process" Ph.D. dissertation, N° d'order 073-2007., Lyon-1 University, Lyon, France, June 2007.

BIOGRAPHIES

Ahmed Baaiu was born in Misurata, Libya, on March 1976. In 1998 he received his B. Sc from the Department of Electrical and Computer Engineering, Faculty of Engineering, NASSER university, Alkhoms, Libya. His B. Sc project was focused on the Analysis of control systems with a delay time effects. In 2003 he received his M.Sc. from Lyon 1 university, Lyon, France. His master subject was entitled "Studying a reference model behavior using the Ramadge and Whonam theory for the synthesis of control laws for complex discrete events systems". In 2007 he received his Ph.D. in automatic control (section 61: computer and control) from Lyon 1 university, Lyon, France. His Ph.D. subject was entitled "Hamiltonian Approach for modelling, estimation and control of a separation process". On September 2007 he appointed as a lecturer at the department of Electrical and Electronic Engineering at the Faculty of Engineering at Misurata University, Misurata, Libya. From February 2011 he promoted to be an Assistant Professor at Misurata University, Misurata, Libya.

Ismail Albatrookh was born in Misurata, Libya, on 6th of June1977. In 2000 he received his B. Sc from the Department of Electrical Engineering, Faculty of Engineering, OMAR ALMUKHTAR university, ELBEIDA, Libya. His B. Sc project was focused on study simplified model of A direct sequence spread spectrum communication system. In 2008 he received his M.Sc. from Tripoli university, Tripoli, Libya. His master subject was entitled"Using Artificial Neural Network to Control an Autonomous Underwater Vehicle". On September 2009 he appointed as a lecturer Assistant at the department of Electrical and Electronic Engineering at the Faculty of Engineering at Misurata University, Misurata, Libya. From August 2013 he promoted to be an Lecturer at Misurata University, Misurata, Libya.